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AERODYNAMICS NOTE 399

ESTIMATION OF AIRCRAFT DYNAMIC STATES AND INSTRUMENT SYSTEMATIC ERRORS FROM FLIGHT TEST MEASUREMENTS USING THE CARLSON SQUARE ROOT FORMULATION OF THE KALMAN FILTER.

by

C. A. MARTIN

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SUMMARY

The development of a procedure for estimating aircraft dynamic states and instrument systematic errors from flight test measurements is described. The method has particular application in non-steady performance estimation for reconstructing aircraft flight path and in the estimation of aerodynamic characteristics using the "equation error" parameter estimation method. The state estimator can be extended to determine systematic measurement errors in the recorded data, giving a set of data which is compatible according to the kinematic equations which relate the measurements. The effectiveness of the procedures cannot be specified in a general way, since the results depend upon the representation of the input and output noise characteristics and on the choice of initial conditions for a given problem.

This note has been written to allow users to apply the state estimation procedure to practical problems. A description of the Carlson Square Root Filter and its application to the kinematic equations of aircraft motion is given. The documentation of the computer program for state estimation is also presented.

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16. ABSTRACT

The development of a procedure for estimating aircraft dynamic states and instrument systematic errors from flight test measurements is described. The method has particular application in non-steady performance estimation for reconstructing aircraft flight path and in the estimation of aerodynamic characteristics using the "equation error" parameter estimation method. The state estimator can be extended to determine systematic measurement errors in the recorded data, giving a set of data which is compatible according to the kinematic equations which relate the measurements. The effectiveness of the procedures cannot be specified in a general way, since the results depend upon the representation of the input and output noise characteristics and on the choice of initial conditions for a given problem.

This note has been written to allow users to apply the state estimation procedure to practical problems. A description of the Carlson Square Root Filter and its application to the kinematic equations of aircraft motion is given. The documentation of the computer program for state estimation is also presented.

CONTENTS

NOTATION

A	state matrix
a_x, a_y, a_z	longitudinal, lateral and normal components of acceleration, m/sec^2
B	matrix of input coefficients
b_y	constant bias error in variable y
$E\{\ \}$	expected value
$f(\ )$	function which represents model
g	acceleration due to gravity, m/sec^2
H	$\partial h(\ )/\partial X$ evaluated along reference trajectory
h	altitude, m
$h(\ )$	function which represents measurement system
I	identity matrix
K	Kalman filter gain
k	discrete time index
m	number of output measurements at each time
n	number of system states
n_v	output measurement noise vector
n_w	input measurement noise vector
n_y	measurement noise associated with variable y
$P(k+1)$	state covariance matrix
$P(t)$	represents state covariance matrix in a continuous system
p, q, r	roll, pitch and yaw velocities, rad/sec
$Q(t)$	represents process noise covariance matrix in a continuous system
$Q(k+1)$	process noise covariance matrix
$R(t)$	represents measurement noise covariance matrix in a continuous system
$R(k+1)$	measurement noise covariance matrix
$S(k)$	square root of state covariance matrix
t	time, sec
$v(k+1)$	measurement noise vector
V	true airspeed, m/sec
u, v, w	longitudinal, lateral and vertical components of airspeed m/s
$W(k+1)$	intermediate square root matrix for state covariance
X	state vector
X_A	augmented state vector

x_b, y_b, z_b	linear position co-ordinates of aircraft, m
x_α, x_β	longitudinal position of incidence and sideslip sensors, m
Y	output vector
y	output variable
y_α	lateral position of incidence sensor, m
Z_E	measurement vector
z_β	vertical position of sideslip sensor, m
z	measurement variable
α_v	incidence angle sensor measurement, rad
β_v	sideslip angle sensor measurement, rad
Γ	input noise matrix
η	input vector
Θ	vector of unknown parameters
θ	pitch angle
λ_y	scale factor error in variable y
ν	vector of measurement residuals
σ^2	variance
τ	time lag
Φ	state transition matrix for linearised equations
ϕ	roll angle, rad
$\chi(k+1)$	covariance of the input measurement noise
ψ	yaw angle, rad
$\omega(k+1)$	process noise vector

Subscripts

E	measured quantity
i	index in matrices
j	index in matrices
0	zero time

Superscripts

T	matrix transpose
-1	matrix inverse
$+$	denotes values updated by measurements

Mathematical notation

$\cdot$	over symbols denotes derivative with respect to time
$\hat{}$	over symbols denotes estimated value
Δ	incremental value
(k)	denotes the value at time $t = k\Delta t$
$(k+1, k)$	denotes prediction of values at the $(k+1)$th time based on values at the (k)th time

1. INTRODUCTION

An important function of the Aircraft Behaviour Studies—Fixed Wing Group of Aerodynamics Division is the determination of aircraft aerodynamic characteristics from flight test measurements. To extend the present analysis capability, a method for aircraft state estimation has been developed under Task D.S.T. 79/105.

Significant advances have been made since 1960 in the analysis of aircraft flight test measurements for the determination of aircraft handling and performance characteristics. Many of the techniques developed during this period are now used as standard analyses methods in flight test centres. One of these techniques, for the estimation of aircraft states and instrument systematic errors, is described in this Note. The technique has been programmed on a digital computer, and is intended to augument methods currently used at the Aeronautical Research Laboratories for aircraft parameter estimation.

For the accurate estimation of aircraft handling and performance characteristics, a comprehensive knowledge of the aerodynamic forces acting on the aircraft is required. A comparison of wind tunnel and flight test methods shows the contrasting nature of the two approaches and illustrates the particular problems associated with making flight test measurements. When testing aircraft models in a wind tunnel, it is possible to measure the forces required directly, and in many cases, to a high degree of accuracy. In addition, it is possible to determine the contributions made to the total forces by individual components of the model. Unfortunately, there are limitations in the accuracy with which the flow field of the full scale aircraft can be represented and this leads to inaccuracy in the prediction of full scale aircraft characteristics. With the flight testing of aircraft, it is only practicable to determine the total aerodynamic forces, and these are calculated indirectly from measurements of aircraft motion. The aerodynamic forces are then determined mathematically from a knowledge of the equations (or mathematical model) relating the forces and motion of the aircraft. Often, the motion variables which can be measured adequately are not the variables, called state variables, which characterise the system state or condition at any instant. In these cases additional algebraic relations are required in the model. During the analysis, inaccuracies within the measurement can lead to large inaccuracies in the estimated forces. Until recent years, simple mathematical models have been used for flight test analysis and the flight test manoeuvres have been limited to a number of steady flight conditions, for example, steady level or steady turning flight. By assuming simple flight conditions, analysis errors are reduced, but the difficulty of satisfying the assumption of steady flight in practice, introduces additional modelling errors. The complete description of forces acting on an aircraft is taken as the aggregate of the results from a number of different tests.

The flight test methods which have recently been developed address the general motion of an aircraft and therefore use a more comprehensive mathematical model to determine the aerodynamic forces. The general approach to flight test measurement is known as system identification and grew from developments in modern control theory, from existing theories of statistical inference and was made possible by the availability of high speed digital computers and modern flight data recording systems. System identification techniques permit a greater number of aerodynamic quantities to be estimated simultaneously, reduce test time significantly compared with steady state testing techniques, generally produce improvements in the accuracy of estimated results, and provide information on the accuracy of the estimates.

Identification has been defined as the determination, based on input and output measurements of a system (or model) to which the system under test is equivalent, and involves the following three stages. Firstly, characterisation of a mathematical model to describe the system. Secondly, estimation of the values of parameters used in the model; this stage may require estimation of the system states from the measured inputs and outputs. Finally, it is necessary to verify that the results obtained are consistent with the known physical characteristics of the system. As the range of applications widens increasing research effort is being given to the

important first stage of model characterisation. A number of parameter estimation techniques, required for stage two, have been developed and used successfully on a wide range of flight vehicles. For certain applications state estimation methods have also been developed and used. In this note, a digital computer program for system state estimation is described. The method has been developed to augment the methods currently used for system identification.

The estimation of system states in the presence of process and measurement noise is achieved essentially by the Kalman Filter first proposed in 1960 in Ref. 1. The filter was derived assuming a linear (not necessarily stationary) dynamic system in which the system outputs and system states are also linearly related.

Following the publication of Ref. 1, the concepts of prediction, filtering and smoothing were introduced. These headings describe respectively the estimation of the state from measurements made prior to, coincident with or subsequent to the time considered. Different forms of smoothing can be used and these can be derived from the Kalman Filter equations. Augmenting the Kalman Filter with a smoothing process will improve the accuracy of the state estimation but will introduce additional computation. The selection of an appropriate smoothing method for the estimation of aircraft dynamic states and instrument bias errors is discussed in Section 3. State estimation of nonlinear dynamic systems can be realised, as discussed in Refs 2 and 3, by linearising the system state and measurement equations around the best estimates of states at each data point. Estimation of measurement bias errors or system parameters can also be realised by the filter by augmenting the state vector with the unknown parameters. When applied to nonlinear equations, the filter, is known as the Extended Kalman Filter. Refs 4 and 5 have demonstrated that Square Root Filters have better numerical properties and give more accurate results than the Kalman Filter. The Square Root Filter is essentially a Kalman Filter in which the square root of the state covariance matrix, rather than the matrix itself, is propagated.

State estimation methods employing Kalman and Square Root Filters have been used successfully in the estimation of aircraft performance characteristics, Refs 3 and 6, and also for estimating aircraft parameters from flight test data containing significant process noise, Refs 7 and 8. In certain cases, Refs 2 and 9, the Extended Kalman Filter has been used to estimate additional parameters such as systematic measurement errors and aerodynamic coefficients. However, as noted in Ref. 10, the use of the Extended Kalman Filter for estimating parameters has not received general acceptance.

In this Note a state estimation program is described which can be used for estimating both aircraft dynamic states and systematic measurement errors. Prior to running the program a selection can be made which allows state estimation of the full set of motion variables or of the following subsets: longitudinal motion with speed variable; longitudinal motion with speed constant and lateral motion. A further selection introduces the augmented state vector to permit estimation of specified systematic measurement errors. The program can also be used for reconstructing certain flight records which may have been lost due to instrumentation malfunction or signal limiting.

The model used for state estimation comprises the nonlinear kinematic equations describing aircraft motion as formulated in Ref. 2. Measurements of aircraft, position, velocity and acceleration provide the inputs and outputs to the model. The model states which are augmented by systematic measurement errors are estimated using the Carlson Square Root Filter developed in Ref. 4.

As discussed above the state estimation program can be applied to a number of different flight test analysis tasks. For each application the results depend upon the selection of initial conditions and of the statistics of the process and measurement noise. Methods for selecting these values are still under development, e.g. Ref. 13. For these reasons it is not possible to evaluate the effectiveness of the procedures in a general way. The purpose of this Note is, firstly, to describe the application of a square root filter to the determination of aircraft dynamic states using the nonlinear kinematic equations of aircraft motion; and, secondly, to document and describe the operation of the state estimation computer program so that it can be used and developed for the practical applications previously described.

Section 2 of this Note discusses the formulation of the model, which is based on the kinematic equations of aircraft motion and Section 3 describes the identification technique employed. Section 4 describes the organisation of the computer program and Section 5 presents

notes on the verification of the correct operation of the computer program. In Section 6 notes are provided on the use of the computer program.

2. MATHEMATICAL MODEL

The mathematical model used for state estimation comprises the set of nonlinear kinematic equations relating the position, velocity and acceleration of a moving rigid body with reference to a set of flat earth axes. The nine equations for body linear velocity, Euler angles and linear position, which can be considered to be exact, are:—

$$\dot{u} = -qw + rv + a_x - g \sin \theta \tag{1}$$

$$\dot{v} = -ru + pw + a_y + g \cos \theta \sin \phi \tag{2}$$

$$\dot{w} = qu - pv + a_z + g \cos \theta \cos \phi \tag{3}$$

$$\dot{\phi} = p + q \sin \phi \tan \theta + r \cos \phi \tan \theta \tag{4}$$

$$\dot{\theta} = q \cos \phi - r \sin \phi \tag{5}$$

$$\dot{\psi} = q \sin \phi / \cos \theta + r \cos \phi / \cos \theta \tag{6}$$

$$\begin{aligned} \dot{x}_b = {} & u \cos \theta \cos \psi + v(\sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi) \\ & + w(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{y}_b = {} & u \cos \phi \sin \psi + v(\sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi) \\ & + w(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \end{aligned} \tag{8}$$

$$\dot{z}_b = -u \sin \theta + v \cos \theta \sin \phi + w \cos \theta \cos \phi \tag{9}$$

Formulation of the state equations comprising the set of state variables, input variables and output variables follows that given in Ref. 2.

The input variables are taken to be the aircraft centre of gravity acceleration components a_x, a_y and a_z and the body rotational rate components p, q and r.

The output variables are flight path velocity V, sideslip sensor angle β_v, incidence sensor angle α_v, the aircraft Euler angles ϕ, θ, and ψ and aircraft altitude $h = -z_b$. For flight dynamic testing, the horizontal displacements x_b and y_b are generally of no interest.

The system state variables are the aircraft velocity components u, v, and w, the aircraft Euler angles ϕ, θ and ψ and aircraft altitude h.

Providing that the input measurements are available, then by straightforward integration of Eqns (1) to (9) the state variables X, and hence the output variables Y can be estimated. This estimate can be improved by use of the Kalman Filter if one or more of the output variables are available. The filter provides an optimal estimate $\hat{X}$ of the state X based on the noise statistics of the state variables, including input noise characteristics, and of the output measurements. In addition to the random noise, the input and output measurements also contain systematic errors, due to instrument bias and scale errors. It is assumed therefore, that each measurement can be expressed as:

$$z = (1 + \lambda_y)y + b_y + n_y \tag{10}$$

where y is the true value of the output;

λ_y is an unknown scale factor error;

b_y is an unknown bias error;

n_y is the measurement noise.

The term λ_y can be used to represent an error in the first order coefficient of a transducer calibration curve. It can also be used to account for errors in the measurement of airspeed (V), sideslip angle (β) and incidence angle (α) because of differences between the local flow at the sensors and the free stream flow. In the formulation of the equations given below, it will be assumed that the instrument calibrations are accurate except for an unknown bias error b_y. The errors in V, β, and α due to local flow conditions at the sensors will be assumed to be linear

and will be represented by λ_v, λ_α, λ_β. The total number of unknown parameters, comprising the instrument bias errors and scale factor errors, is sixteen.

Replacing the input variables in the state equations by their measured values, Z_E, gives the following set of equations:

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} 0 & r_E - b_r & -(q_E - b_q) & 0 \\ -(r_E - b_r) & 0 & p_E - b_p & 0 \\ q_E - b_q & -(p_E - b_p) & 0 & 0 \\ \sin\theta & -\cos\theta\sin\phi & -\cos\theta\cos\phi & 0 \end{bmatrix} \times \begin{bmatrix} u \\ v \\ w \\ h \end{bmatrix}$$

$$+ \begin{bmatrix} -g\sin\theta & +a_{x_E} & -b_{a_x} \\ g\cos\theta\sin\phi & +a_{y_E} & -b_{a_y} \\ g\cos\theta\cos\phi & -a_{z_E} & +b_{a_z} \\ 0 & & \end{bmatrix} + \begin{bmatrix} 0 & w & -v & 0 \\ -w & 0 & u & 0 \\ v & -u & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} n_p \\ n_q \\ n_r \\ 0 \end{bmatrix} - \begin{bmatrix} n_{a_x} \\ n_{a_y} \\ -n_{a_z} \\ 0 \end{bmatrix} \tag{11a}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \times \begin{bmatrix} p_E & -b_p & +n_p \\ q_E & -b_q & +n_q \\ r_E & -b_r & +n_r \end{bmatrix} \tag{11b}$$

Positive a_z is defined in the negative z direction to be consistent with flight test and instrumentation convention.

The output equations take the form:

$$\left.\begin{aligned} V &= (1 + \lambda_v)\sqrt{(u^2 + v^2 + w^2)} + b_v \\ \beta_v &= (1 + \lambda_\beta)\tan^{-1}\left[\frac{v + (r_E - b_r)x_\beta - (p_E - b_p)z_\beta}{u}\right] + b_\beta \\ \alpha_v &= (1 + \lambda_\alpha)\tan^{-1}\left[\frac{w - (q_E - b_q)x_\alpha + (p_E - b_p)y_\alpha}{u}\right] + b_\alpha \\ \begin{bmatrix} \phi \\ \theta \\ \psi \\ h \end{bmatrix}_{\text{OUTPUT}} &= \begin{bmatrix} \phi \\ \theta \\ \psi \\ h \end{bmatrix}_{\text{STATE}} + \begin{bmatrix} b_\phi \\ b_\theta \\ b_\psi \\ b_h \end{bmatrix} \end{aligned}\right\} \tag{12}$$

Substituting the measured values into the state equations introduces process noise, which in practice, is both non-stationary and cross correlated. To permit application of the Kalman Filter, it is assumed that the cross correlation is zero and that the process noise is purely random, with zero mean but with time varying covariance. It is also assumed that the output measurements do not include the effects of atmospheric turbulence. In addition the random errors in the measured rotational rates have been neglected in the corrections for sensor location in the β_v and α_v output equations.

The general form of the system equations can be written:

state equation $\quad \dot{X} = f(X, \eta_E, \Theta, t) + \omega(t) \quad$ (13)

output equation $\quad Y = h(X, \eta_E, \Theta) \quad$ (14)

measurement equation $\quad Z_E = Y + n_v(t) \quad$ (15)

$\omega(t)$ and $n_v(t)$ denote the system process noise and the output measurement noise respectively. For the system equations under consideration it is shown in Appendix A that $\omega(t)$ is due predominantly to the input measurement noise $n_w(t)$.

For the complete set of equations, the system vectors are:

state vector $\quad X^T = [u, v, w, h, \phi, \theta, \psi]$

input vector $\quad \eta_E{}^T = [a_{x_E}, a_{y_E}, a_{z_E}, p_E, q_E, r_E]$

output vector $\quad Y^T = [V, \beta_v, \alpha_v, h, \phi, \theta, \psi]$

measurement vector $Z_E^T = [V_E, \beta_{v_E}, \alpha_{v_E}, h_E, \phi_E, \theta_E, \psi_E]$

vector of unknown parameters

$$\Theta^T = [b_{a_x}, b_{a_y}, b_{a_z}, b_p, b_q, b_r, b_v, b_\beta, b_\alpha, b_h, b_\phi, b_\theta, b_\psi, \lambda_v, \lambda_\beta, \lambda_\alpha]$$

A reduced set of equations and number of states to be estimated can be selected, depending on the input measurements available.

3. IDENTIFICATION TECHNIQUE

Estimation of the states and unknown parameters of the dynamic system described by Eqns (13), (14) and (15) is achieved by using a square root formulation of the Kalman Filter, with an extension to account for system nonlinearities.

3.1 The Kalman Filter

For a linear system, represented by a set of discrete equations, an estimation of the state and state covariance can be obtained from the following equations. The discrete state equation for the system is:

$$X(k+1) = \Phi(k+1, k)\, X(k) + \omega(k+1) \qquad k = (0, 1, 2, \ldots) \tag{16}$$

where $X(k)$ is the n by 1 state vector at the kth time;

$\Phi(k+1, k)$ is the n by n state transition matrix from the kth to the $(k+1)$th time;

$\omega(k+1)$ is the n by 1 vector of process noise having a constant value between the kth and $(k+1)$th time.

It is assumed that $\omega(k+1)$ is a sample from a purely random noise process with zero mean and covariance $Q(k+1)$. Associated with the state equations are the linearised measurement equations.

$$\begin{aligned} Z_E(k+1) &= Y(k+1) + v(k+1) \\ &= H(k+1)\, \hat{X}(k+1) + v(k+1) \qquad (k = 0, 1, 2, \ldots) \end{aligned} \tag{17}$$

where $Z_E(k+1)$ is the m by 1 measurement vector;

$Y(k+1)$ is the m by 1 output vector;

$H(k+1)$ is the m by n output matrix;

$v(k+1)$ is the m by 1 vector of measurement noise at the $(k+1)$th time.

The noise vector $v(k+1)$ is assumed to be a sample from a purely random measurement noise process $n_v(t)$ and has zero mean and diagonal covariance matrix $R(k+1)$.

To determine an optimal estimate for $X(k)$, denoted $\hat{X}(k)$, it is necessary to provide an initial estimate of the state $X(0)$ and an initial value of the state covariance matrix $P(0)$.

With the assumptions given above for the process noise $Q(k+1)$, the Kalman Filter propagates the estimated state $\hat{X}(k+1)$ and state covariance $P(k)$ to the $(k+1)$th time according to the equations:

$$\hat{X}(k+1) = \Phi(k+1, k)\, \hat{X}(k) \tag{18}$$

$$P(k+1) = \Phi(k+1, k)\, P(k)\, \Phi(k+1, k)^T + Q(k+1) \tag{19}$$

superscript T denotes transpose.

The derivation of the process noise matrix $Q(k+1)$ is given in Appendix A.

At the $(k+1)$th time the information contained in the measurement vector $Z_E(k+1)$ is incorporated by computing the optimal gain matrix $K(k+1)$ according to the following equation

$$K(k+1) = P(k+1)\, H(k+1)^T\, [H(k+1)\, P(k+1)\, H(k+1)^T + R(k+1)]^{-1} \tag{20}$$

superscript -1 denotes matrix inversion.

The derivation of the measurement noise matrix $R(k+1)$ is given in Appendix A.

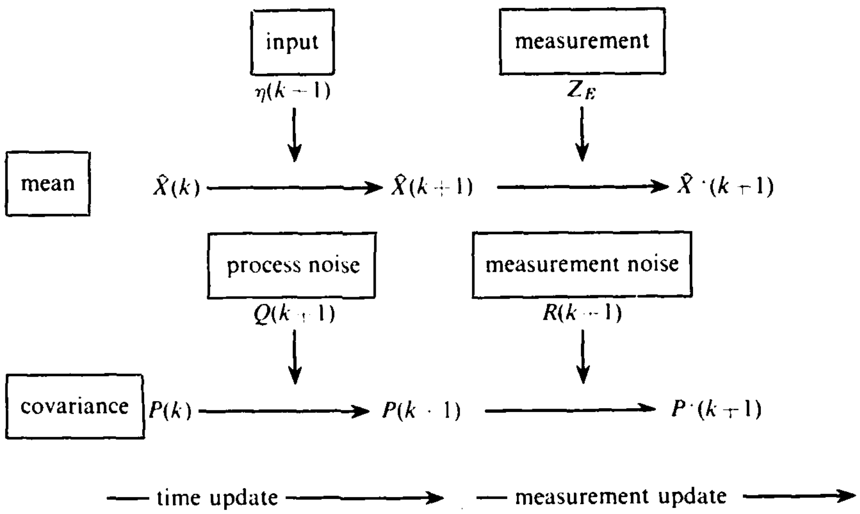
By using the optimal gain, updated values of the state and covariance are calculated from:

$$\hat{X}^{+}(k+1) = \hat{X}(k+1) + K(k+1)\,[Z_E(k+1) - H(k+1)\,\hat{X}(k+1)] \tag{21}$$

$$P^{+}(k+1) = P(k+1) - K(k+1)\,H(k+1)\,P(k+1) \tag{22}$$

superscript $+$ denotes values updated by incorporating the measurements at the $(k+1)$th time.

The procedure described by Eqns 18 to 22 is illustrated in the following diagram:



The accuracy of the estimates from the Kalman Filter depends strongly upon the values specified for the process noise covariance $Q(k+1)$ and the measurement noise covariance $R(k+1)$. To assist in selecting appropriate values, use can be made of the output residuals, ν which are defined as the difference between the measured variables, Z_E, and the predicted measurements Y. That is: $\nu(k+1) = Z_E(k+1) - Y(k+1)$. (23)

As discussed in Ref. 2, with the correct values of process and measurement noise, and assuming that the system is modelled correctly, then the residuals should approach a random Gaussian white sequence with zero mean and a variance consistent with the values calculated from the filter, Eqn (20), i.e.:

$$E\{\nu(k+1)\,\nu(k+1)^T\} = H(k+1)\,P(k+1)\,H(k+1)^T + R(k+1) \tag{24}$$

Since the matrix $P(k+1)$ includes the effect of the process noise, then a comparison of the calculated residuals from Eqn (23) and their expected variance from Eqn (24) can be used to adjust the initial choice of process and measurement noise, which are not known *a priori*.

3.2 The Carlson Square Root Formulation of the Kalman Filter

A practical problem in the use of Eqn (22) is discussed in Refs 4 and 5. By definition, $P(k)$ is positive definite since $P(k) = E\{\hat{X}(k)\,\hat{X}(k)^T\}$ and Eqns (18) to (22) will propagate a positive definite matrix; however, because of the finite word length of computers, the matrix subtraction in Eqn (22) often yields a non-positive unsymmetric matrix after propagation through a number of time points. This results in a non-optimal estimate for $\hat{X}(k)$ which can diverge from realistic values. In Ref. 4 Carlson has developed an algorithm for propagating the square root of the state covariance matrix $S(k)$ where

$$S(k)\,S(k)^T = P(k) \tag{25}$$

This procedure ensures that $P(k)$ cannot be unsymmetric or indefinite. The algorithm has been evaluated in Ref. 5 and has been shown to give greater precision than the Kalman Filter Eqns (18) to (22).

The Carlson Filter has been formulated to minimise run time and computer storage by ensuring that the square root of the state covariance matrix $S(k)$ remains triangular during measurement updating. By careful programming, as detailed in Ref. 4, the computational speed is shown to be better than most square root filters and comparable with the speed of the Kalman Filter. Details of the computer program for the Carlson Filter are given in Section 4.

In the Carlson Filter, Eqn (19), which is used for the time updating of $P(k)$, is replaced by the following equations:

$$W(k+1) = \Phi(k+1, k)\, S^{+}(k) \tag{26}$$

$$S(k+1) = [W(k+1)\, W(k+1)^T + Q(k+1)]^{\frac{1}{2}} \tag{27}$$

where $[\ ]^{\frac{1}{2}}$ denotes formation of the square root matrix.
In the Carlson Filter a Choleski decomposition is used to generate $S(k+1)$ in upper triangular form. This can be accomplished with a finite procedure since the state covariance matrix is positive definite.

In the computer program, the upper triangular matrix $S(k+1)$ is stored columnwise and accessed as a vector. $\Phi(k+1)$, $W(k+1)$ and $Q(k+1)$, although square n by n matrices, are partitioned and only the segments containing non-zero elements are stored.

The equations for measurement updating in the Kalman Filter, Eqns (20) to (22) are replaced by the following equations:

$$f(k+1) = S(k+1)^T H_j(k+1)^T \tag{28}$$

$$\alpha_0 = R_{jj}(k+1) \tag{29}$$

$$b_0(k+1) = 0 \tag{30}$$

$$\alpha_i = \alpha_{i-1} + f_i^2(k+1) \tag{31}$$

$$a_i = \left(\frac{\alpha_{i-1}}{\alpha_i}\right)^{\frac{1}{2}} \tag{32}$$

$$c_i = \frac{f_i(k+1)}{(\alpha_{i-1}\,\alpha_i)^{\frac{1}{2}}} \tag{33}$$

$$b_i(k+1) = b_{i-1}(k+1) + S_i(k+1) f_i(k+1) \tag{34}$$

$$S_i(k+1) = S_i(k+1) a_i - b_{i-1}(k+1) c_i \tag{35}$$

Cycle $i = 1, n$ for each measurement (j)

$$\Delta Z(k+1) = Z_{E_j}(k+1) - H_j(k+1)\,\hat{X}(k+1) \tag{36}$$

$$\hat{X}(k+1) = \hat{X}(k+1) + \frac{b_n(k+1)\,\Delta Z(k+1)}{\alpha_n} \tag{37}$$

Cycle $j = 1, m$ at each time interval

$$\hat{X}^{+}(k+1) = \hat{X}(k+1) \tag{38}$$

$$\hat{S}^{+}(k+1) = S(k+1) \tag{39}$$

α_0, α_i, a_i and c_i for $(1 \leqslant i \leqslant n)$ are scalars;

$b_0(k+1)$ and $b_i(k+1)$ for $(1 \leqslant i \leqslant n)$ are n dimensional vectors;

$f_i(k+1)$ is the ith component of $f(k+1)$;

$S_i(k+1)$ is the ith column of $S(k+1)$;

$H_j(k+1)$ is the jth row of $H(k+1)$;

$R_{jj}(k+1)$ is the jth diagonal element of $R(k+1)$;

$Z_{E_j}(k+1)$ is the jth component of $Z_E(k+1)$;

$\Delta Z(k+1)$ is a scalar.

The updated matrix $S^{+}(k+1)$ remains upper triangular but this does not eliminate the need for the Choleski decomposition unless the process noise $Q(k+1)$, which is added in Eqn (27), is zero.

3.3 Estimation of System Parameters

To extend the estimation technique to include the systematic error terms in Eqns (11) and (12), the system state vector X is augmented with the vector of unknown parameters Θ giving the augmented state vector

$$X_A = \begin{bmatrix} X \\ \hdashline \Theta \end{bmatrix} \tag{40}$$

This extension introduces non-linearities into the system equations which already have intrinsic non-linearities from the kinematic equations. To estimate the augmented state vector for the set of non-linear equations, the Extended Kalman Filter is used. In this approach the system state and output equations are linearised about the best estimate of the state at each data point as described in Appendix B.

The state and output equations [Eqns (13) and (14)] are changed to:

$$\dot{X}_A = \begin{bmatrix} f(X_A, \eta_E, t) \\ \hdashline 0 \end{bmatrix} + \begin{bmatrix} \omega(t) \\ \hdashline 0 \end{bmatrix} \tag{41}$$

$$Y = h(X_A, \eta_E) \tag{42}$$

Time updating of the system state [given by Eqn (18) for the linear case] is now carried out by numerical integration of Eqn (41).

The state transition matrix required for updating of the state covariance matrix by Eqn (19) for the Kalman Filter, or by Eqn (26) for the Carlson square root formulation, is obtained by linearising Eqn (41) as shown in Appendix B.

From Eqn (41) it can be seen that the appended state variables Θ and their associated covariances will remain unaltered during time updating. This will have the following implications on the choice of initial conditions. Measurement updating in the Kalman Filter is carried out by Eqns (20) to (22).

Substituting Eqn (20) into Eqn (22) gives:

$$P^{+}(k+1) = P(k+1) - P(k+1)\,H(k+1)^T[H(k+1)\,P(k+1)\,H(k+1)^T + R(k+1)]^{-1}\,H(k+1)\,P(k+1) \tag{43}$$

which can be written

$$P^{+}(k+1)^{-1} = P(k+1)^{-1} + H(k+1)^T\,R(k+1)^{-1}\,H(k+1) \tag{44}$$

It can be seen from Eqn (44) that the state covariance matrix after measurement update is never larger than the value before update, since $H(k+1)^T\,R(k+1)^{-1}\,H(k+1)$ is at least a positive semi-definite matrix. Thus the act of measurement on the average, never increases the uncertainty in the knowledge of $X(k)$. Since the covariance of the unknown parameters remain unaltered during the time updating stage, it is necessary to choose initial values for these covariances which are greater than zero. Selection of initial conditions for the filter is discussed in Section 6.

3.4 State Smoothing

The "filtering process" which, by definition, calculates the best estimate of the state at a given time from all the measurements up to that time, can be augmented with a "smoothing process". The smoothing process enables estimates $\hat{X}(k, k+1)$ and $P(k, k+1)$ to be made; that is, the information added at the $(k+1)$th time can be used to give an estimate at the kth time, and if required, at all previous times. In Ref. 2 single-stage smoothing and local iteration of the state estimate $\hat{X}(k)$ was carried out, in an attempt to reduce the bias which is inherent within the Extended Kalman Filter. In addition, fixed point smoothing was used to update initial conditions after each pass through the filter. A smoothing procedure has not been included in the program described in this Note. The program can be developed to include the appropriate smoothing procedure for each application.

4. DESCRIPTION OF THE STATE ESTIMATION COMPUTER PROGRAM

Effort has been made in the computer program to take advantage of matrix symmetry and triangularity in the updating of the square root covariance matrix $S(k)$ and of general blocks of zeroes in the state transition matrix, to minimise computation and storage requirements. A summary of operations performed by the program is given in Fig. 1.

Only the storage and computation of the first i entries in each column of $S_i(k)$ is required. In addition, it is only necessary to store the $\Phi_{11,12}$ elements of the state transition matrix since the identity matrix Φ_{22} is implicit in Step 9.4 of Fig. 1. Numerical integration of the state variables is carried out by the variable step size integration routine described in Ref. 11. The calculation of the state matrix A is carried out during the integration step; when only a subset of the equations, for example the kinematic equations for longitudinal motion, or when a reduced number of states are to be estimated, only the associated equations are computed and integrated. Similarly during measurement updating, calculations are only performed for the set of output measurements specified in the program set-up data.

5. VERIFICATION OF COMPUTER PROGRAM

The integrity of the computer program has been checked by successively expanding the size of the system equations from a simple set which is amendable to analytic solution. Initially the prediction of the state and state covariances for a second-order system without measurement updating, were compared with the analytic solutions given in Ref. 12. Measurement updating was then added and the results were compared with the numerical solutions also given in Ref. 12.

Using the Kalman Filter equations and matrix routines verified in the two degrees of freedom filter, a state estimation program was written with the kinematic equations for the motion of a body [Eqns (11) and (12)] as the system model. The same system of equations was also programmed using the Carlson square root formulation of the Kalman Filter, and particular effort was made to minimise computation and storage requirements, as discussed in Section 4. The results from the Kalman Filter were used to verify the scalar algorithms employed in the Carlson Filter. Finally the Carlson Filter was expanded to include the estimation of systematic errors in the input and output measurements.

Because of the large number of permutations of model complexity and of states which can be estimated, and because of the dependence of the estimates on the set-up data, the general performance of the filter has not been assessed. Refs 3 and 6 have demonstrated satisfactory results for the estimation of the state variables of longitudinal motion of an aircraft and estimation of three instrument bias errors. In Ref. 2, good results were claimed for the estimation of the lateral response of aircraft motion with five states and ten parameters. So far satisfactory estimation of the full set of seven states and twenty-three parameters from measured data has not been demonstrated.

6. NOTES ON THE USE OF THE STATE ESTIMATION COMPUTER PROGRAM

Set up information for the state estimation program called FILTER is prepared by running program CHOICE. A copy of the set up data is presented in Appendix C. Experience with the Kalman Filter shows that the accuracy of the filter, particularly when the augmented state vector is included, is critically dependent upon the values specified in the set up data for the initial state conditions and the input/output measurement noise.

Stage 1 of CHOICE requires selection of the values for the input and output measurement noise variances. Initial estimates are made from a knowledge of the precision of the instrumentation system producing the measurements. These estimates can be refined by an analysis of the residuals following state estimation, by using Eqns (23) and (24).

Stage 2 of CHOICE requires specification of the input and output measurement channels to be used. Depending upon the particular input measurements available, four different combinations of the filter equations are available as shown in Table 1. If a particular measurement is not available, the status of that channel is set at zero, and its steady state value has to be

TABLE 1

Possible Selections of States to be Estimated

Value	Longitudinal Equations—Speed and Height Constant	Longitudinal Equations—Speed and Height Variable	Lateral Equations—Speed and Height Constant	Complete Longitudinal and Lateral Motion
	Input Measurements Required			
a_x		●		●
a_y			●	●
a_z	●	●		●
p			●	●
q	●	●		●
r			●	●
	States which can be estimated			
u		●		●
v			●	●
w	●	●		●
h		●		●
ϕ			●	●
θ	●	●		●
ψ			●	●
b_{a_x}		●		●
b_{a_y}			●	●
b_{a_z}	●	●		●
b_p			●	●
b_q	●	●		●
b_r			●	●
b_v		●		●
b_β			●	●
b_α	●	●		●
b_h		●		●
b_ϕ			●	●
b_θ	●	●		●
b_ψ			●	●
λ_v		●		●
λ_β			●	●
λ_α	●	●		●
	Outputs which can be calculated			
V	●	●		●
β			●	●
α	●	●		●
h		●		●
ϕ			●	●
θ	●	●		●
ψ			●	●

supplied. If no output measurements are available, the program is run purely as a state predictor. As the number of output channels to be included is increased, the estimation of the state is improved.

Stage 3 of CHOICE allows specification of the particular states to be estimated. Based upon this selection, only those states and associated covariance elements are computed and integrated. During selection of the states to be estimated it is also necessary to specify the initial values of the state covariance matrix $P(k)$. Generally it is sufficient to specify only the variance or leading diagonal of $P(k)$ and for the unaugmented state vector, an arbitrary positive definite choice is satisfactory. When the augmented state vector is included, the initial variances for the additional states should be greater than zero as discussed in Section 3.3. Tests on a second-order system with the state vector augmented by two bias error parameters, showed that satisfactory estimates were only achieved when the initial state variances exceeded their final steady values.

In addition to the set-up data, the program FILTER requires a data file of the measurements in the order specified in the vectors η and Z_E (see Section 2), and also the following information at run-time:

1. RES —an integer which, if greater than 0, instructs the program to calculate residuals;
2. TSTART —specifies start time referred to the beginning of the measurement data file;
3. TTOT —specifies the length of measurement record to be filtered;
4. AQR —specifies the data acquisition rate;
5. No. OF PASSES—an integer which specifies the number of repeated passes through the filter.

After each pass, the initial values of the variables in the extended part of the state vector and the variances of the basic state variables are set to the final values reached in the preceding pass. Initial conditions are used for the basic states and for the variances of the augmented states. Tests with a second-order system, which has the state vector augmented by two bias error parameters, showed that a single pass was sufficient. However, tests on a three degrees of freedom system with the state vector augmented by eight bias states, reported in Ref. 2 showed that continued improvements in the estimates were achieved with up to four passes through the filter.

The output of program FILTER is a file containing the estimated states, and state variances and a file of bias errors. The bias errors are the average of the bias states calculated over the final sixty data points.

The output files from program FILTER are used by program FLIGHT to form a file of compatible data of all the measured quantities and also plotting files of the compatible data, state variables, state variances or residuals if this option has been requested. The compatible data is a file equivalent to the file of measurement data, but with the input variables corrected for bias errors, and the output measurements constructed from the estimated states. An example of the compatible data, estimated using the Carlson Filter together with the measured data for a set of simulated measurements, is plotted in Fig. 2. The input information to program FLIGHT is an integer, greater that 0 if residuals have been calculated, followed by start time, total record time and acquisition rate for the data. A selection of the output files required for plotting can also be made.

When the option to calculate residuals is selected, the computer program FILTER calculates the estimated standard deviation of the residuals from Eqn (24) and stores the results in place of the state variances in channels 7 to 13. The average of the stored values is calculated over the final sixty data points of the time history and is stored with the file of bias errors. When program FLIGHT is informed that residuals have been selected, then, the actual residuals are calculated and plot files of estimated and calculated standard deviations for the full time history are calculated and stored in place of the compatible data and state variance information respectively. A file named RESID is also produced which presents the average standard deviation for the estimated and calculated residuals calculated over the final sixty data points.

When the option to calculate residuals is not selected, the calculations described above are by-passed.

7. CONCLUDING REMARKS

A state estimation computer program has been written for the estimation of aircraft dynamic states and instrument systematic errors from flight test measurements. The program can be used with parameter estimation methods for the determination of aircraft handling and performance characteristics. It has particular application in non-steady aircraft performance estimation, for the reconstruction of aircraft flight path, and in the estimation of aerodynamic characteristics in situations where "equation error" rather than "output error" parameter estimation methods are preferred. The state estimator can be extended to determine measurement bias errors in the recorded data, giving a set of data which is compatible according to the kinematic equations which relate the measurements. Used in this way, the filter has the potential to check the correct operation of channels of flight instrumentation from measured data prior to commencing test manouevres. The state estimator can also be used to reconstruct certain flight records which may have been lost due to instrumentation malfunction or signal limiting. The routines which have been developed for the state estimation program can be used to modify existing parameter estimation programs to include the effects of process noise due, for example, to atmospheric turbulence.

Three subsets of the full equations can be selected which permit separate analysis of longitudinal motion, with or without constant forward speed or analysis of purely lateral motion. The accuracy of the state estimates depend strongly upon the estimated input and output noise characteristics and the initial conditions specified in the set-up data. A rigorous approach to the selection of these quantities has not yet been developed, and so the effectiveness of the procedures can only properly be established through application to particular problems. The present Note provides a description of the method used for estimating the states of a non-linear model using a square root filter, and documents a computer program based on this method. For particular problems, the following possible program developments may be of benefit: firstly, inclusion of a smoothing algorithm, either as a single stage procedure or for updating the initial conditions; secondly, replacement of the variable step size integrator with a fixed-step integrator to reduce program run time; thirdly, development of procedures for the analysis of residuals to aid in the specification of input and output noise statistics.

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APPENDIX A

Calculation of the Process Noise and Measurement Noise Covariance Matrices

In Section 3 the discrete Kalman Filter equations are developed for a linear system represented by the discrete state equation

$$X(k+1) = \Phi(k+1, k)\, X(k) + \omega(k+1) \qquad k = (0, 1, 2, \ldots) \tag{A.1}$$

$\omega(k+1)$ is the process noise and accounts for noise in the input signal, for noise within the process itself, i.e. unmeasured disturbances, and can also be used to account for differences between the model and the actual system. For systems which are accurately modelled and in which the unmeasured disturbances are small, such as the kinematic relations represented by Eqns (13) to (15), it is assumed that the process noise is predominantly caused by noise in the measured inputs $n_\omega(k)$.

Therefore the process noise

$$\omega(k+1) = \Gamma(k+1)\, n_\omega(k+1) \tag{A.2}$$

and the process noise covariance matrix

$$Q(k+1) = \Gamma(k+1)\, \chi(k+1)\, \Gamma^T(k+1) \tag{A.3}$$

$\chi(k+1)$ is a diagonal matrix with elements equal to the variances of the input noise $n_\omega(k+1)$. $\Gamma(k+1)$ is derived in Appendix B.

The measurement noise is simply the noise in the measured outputs

$$v(k+1) = n_v(k+1) \tag{A.4}$$

and the measurement noise covariance matrix $R(k+1)$ is a diagonal matrix with elements equal to the variances of the output measurement noise $n_v(k+1)$.

In the program validation described in Section 5, the solution for a continuous process was approximated by the discrete Kalman Filter. For this case, it is shown in Ref. 12 that the process and measurement noise covariances are determined as follows.

The state equation for the continuous linear system is

$$X = A(t)X + \omega(t) \tag{A.5}$$

where $\omega(t)$ is a Gaussian purely random process

with $E[\omega(t)] = 0$

$$E[\omega(t)\, \omega^T(\tau)] = Q(t)\, \delta(t-\tau)$$

$Q(t)$ is a non-negative definite matrix representing the integral of the random process $\omega(t)$ and $\delta(t-\tau)$ is the Dirac delta function.

Using the discrete approximation given by Eqns (A.1) to (A.3) and choosing to make the state covariance matrix for the discrete system $P(k)$ equivalent to its counterpart $P(t)$ in the continuous system, Ref. 12 shows that

$$\chi(k) = Q(t)/\Delta t \tag{A.6}$$

where Δt is the stepsize.

Similarly it is shown that the measurement noise $v(t)$ which is assumed to be a Gaussian purely random process with

$$E[v(t)] = 0$$

$$E[v(t)\, v^T(\tau)] = R(t)\, \delta(t-\tau)$$

is represented in the discrete system by

$$R(k) = R(t)/\Delta t \tag{A.7}$$

In a practical situation the statistics of the process and measurement noise are not usually known sufficiently accurately prior to state estimation. This can lead to large estimation errors or even to a divergence of the errors. The problem is discussed briefly in Ref. 2 and in more detail in Ref. 13. For the filter discussed in this Note the process and measurement noise is chosen from a comparison of the output residuals, as discussed in Section 3.1.

APPENDIX B

Linearisation of the Kinematic State and Output Equations

For the Extended Kalman Filter, the system state and output equations are linearised about the best estimate of the state at each data point $\hat{X}_A$.

For the kinematic equations, in which the process noise is assumed to be due solely to noise in the input measurements, as discussed in Appendix A, the upper segment of the state equation [Eqn (41)] can be written

$$\dot{X}_A = f(X_A, \eta_E, t) + Bn_\omega(t) \tag{B.1}$$

where B is a matrix of input coefficients from Eqn (11).

Neglecting differences of second order and higher, equation (B.1) can be approximated by the expansion

$$\dot{X}_A = f(\hat{X}_A, \eta_E, t) + \frac{\partial f}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A) + Bn_\omega(t) \tag{B.2}$$

$$= \dot{\hat{X}}_A + \frac{\partial f}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A) + Bn_\omega(t)$$

$$\therefore \qquad (\dot{X}_A - \dot{\hat{X}}_A) = \frac{\partial f}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A) + Bn_\omega(t)$$

i.e.

$$\Delta \dot{X}_A = A\, \Delta X_A + Bn_\omega(t) \tag{B.3}$$

The discrete form of Eqn (B.3) is

$$\Delta X_A(k+1) = \Phi(k+1, k)\, \Delta X_A(k) + \Gamma(k+1)\, n_\omega(k) \tag{B.4}$$

where

$$\Phi(k+1, k) = e^{A\,\Delta t} \tag{B.5}$$

and

$$\Gamma(k+1) = \int_0^{t_1} e^{A(t_1-\tau)}\, B(\tau) \mathrm{d}\tau \tag{B.6}$$

Similarly, the output equation [Eqn (42)] can be approximated as

$$Y = h(\hat{X}_A, \eta_E) + \frac{\partial h}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A) \tag{B.7}$$

where, from Section 2, it is noted that the random errors in the rotational rate inputs have been neglected in the output equations for β_v and α_v.

$$\therefore \qquad Y = \hat{Y} + \frac{\partial h}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A)$$

$$(Y - \hat{Y}) = \frac{\partial h}{\partial X_A}\bigg|_{\hat{X}_A} (X_A - \hat{X}_A)$$

$$\Delta Y = H\, \Delta X_A \tag{B.8}$$

or in discrete form

$$\Delta Y(k+1) = H\, \Delta X_A(k+1) \tag{B.9}$$

As can be seen from Fig. 1 it is only necessary to determine A for the thirteen by thirteen state transition matrix $\Phi_{11/12}(k+1, k)$. Within the computer program, only those elements of A and B which define the states selected for estimation are calculated. Similarly only those elements of H which define the outputs for which measurements are available are calculated.

For the case when all twenty-three states in X_A are to be estimated and when all seven measurement channels in Z_E are available, the following differentials are calculated. The arguments in the notation $A(i,j)$ used below, denote the dependent and independent variables of the partial differentials in the linearised equations respectively. For the linearised state equation:

$$
\begin{aligned}
A(u, v) &= (r_E - b_r) \\
A(u, w) &= -(q_E - b_q) \\
A(u, \theta) &= -g \cos\theta \\
A(u, b_{a_z}) &= -1{\cdot}0 \\
A(u, b_q) &= w \\
A(u, b_r) &= -v
\end{aligned}
$$

$$
\begin{aligned}
A(v, u) &= -(r_E - b_r) \\
A(v, w) &= (p_E - b_p) \\
A(v, \phi) &= g \cos\theta \cos\phi \\
A(v, b_{a_y}) &= -1{\cdot}0 \\
A(v, \theta) &= -g \sin\theta \sin\phi \\
A(v, b_p) &= -w \\
A(v, b_r) &= u
\end{aligned}
$$

$$
\begin{aligned}
A(w, u) &= (q_E - b_q) \\
A(w, v) &= -(p_E - b_p) \\
A(w, \phi) &= -g \cos\theta \sin\phi \\
A(w, \theta) &= -g \sin\theta \cos\phi \\
A(w, b_{a_z}) &= 1{\cdot}0 \\
A(w, b_p) &= v \\
A(w, b_q) &= -u
\end{aligned}
$$

$$
\begin{aligned}
A(h, u) &= \sin\theta \\
A(h, v) &= -\cos\theta \sin\phi \\
A(h, w) &= -\cos\theta \cos\phi \\
A(h, \phi) &= -v \cos\theta \cos\phi + w \cos\theta \sin\phi \\
A(h, \theta) &= u \cos\theta + v \sin\theta \sin\phi + w \sin\theta \cos\phi
\end{aligned}
$$

$$
\begin{aligned}
A(\phi, \phi) &= (q_E - b_q) \cos\phi \tan\theta - (r_E - b_r) \sin\phi \tan\theta \\
A(\phi, \theta) &= (q_E - b_q) \sin\phi/\cos^2\theta + (r_E - b_r) \cos\phi/\cos^2\theta \\
A(\phi, b_p) &= -1{\cdot}0 \\
A(\phi, b_q) &= -\sin\phi \tan\theta \\
A(\phi, b_r) &= -\cos\phi \tan\theta
\end{aligned}
$$

$$
\begin{aligned}
A(\theta, \phi) &= (q_E - b_q) \sin\phi - (r_E - b_r) \cos\phi \\
A(\theta, b_q) &= -\cos\phi \\
A(\theta, b_r) &= \sin\phi
\end{aligned}
$$

$$
\begin{aligned}
A(\psi, \phi) &= (q_E - b_q) \cos\phi/\cos\theta - (r_E - b_r) \sin\phi/\cos\theta \\
A(\psi, \theta) &= (q_E - b_q) \sin\phi \tan\theta/\cos\theta + (r_E - b_r) \cos\phi \tan\theta/\cos\theta \\
A(\psi, b_q) &= -\sin\phi/\cos\theta \\
A(\psi, b_r) &= -\cos\phi/\cos\theta
\end{aligned}
$$

For the linearised output equations:

$$
\begin{aligned}
H(V, u) &= (1+\lambda_V)\, u(u^2+v^2+w^2)^{-\frac{1}{2}} \\
H(V, v) &= (1+\lambda_V)\, v(u^2+v^2+w^2)^{-\frac{1}{2}} \\
H(V, w) &= (1+\lambda_V)\, w(u^2+v^2+w^2)^{-\frac{1}{2}} \\
H(V, b_V) &= 1{\cdot}0 \\
H(V, \lambda_V) &= (u^2+v^2+w^2)^{\frac{1}{2}}
\end{aligned}
$$

denoting $f = [v + (r_E - b_r)\, x_\beta - (p_E - b_p)\, z_\beta]/u$

$$H(\beta, u) = -(1+\lambda_\beta)\frac{1}{(1+f^2)}\frac{f}{u}$$

$$H(\beta, v) = (1+\lambda_\beta)\frac{1}{(1+f^2)}\frac{1}{u}$$

$$H(\beta, b_p) = (1+\lambda_\beta)\frac{1}{(1+f^2)}\frac{z_\beta}{u}$$

$$H(\beta, b_r) = -(1+\lambda_\beta)\frac{1}{(1+f^2)}\frac{x_\beta}{u}$$

$$H(\beta, b_\beta) = 1{\cdot}0$$

$$H(\beta, \lambda_\beta) = \tan^{-1} f$$

denoting $f = [w - (q_E - b_q)\, x_\alpha + (p_E - b_p)\, y_\alpha]/u$

$$H(\alpha, u) = -(1+\lambda_\alpha)\frac{1}{(1+f^2)}\frac{f}{u}$$

$$H(\alpha, w) = -(1+\lambda_\alpha)\frac{1}{(1+f^2)}\frac{1}{u}$$

$$H(\alpha, b_q) = (1+\lambda_\alpha)\frac{1}{(1+f^2)}\frac{x_\alpha}{u}$$

$$H(\alpha, b_p) = -(1+\lambda_\alpha)\frac{1}{(1+f^2)}\frac{y_\alpha}{u}$$

$$H(\alpha, b_\alpha) = 1{\cdot}0$$

$$H(\alpha, \lambda_\alpha) = \tan^{-1} f$$

$$H(h, h) = 1{\cdot}0$$

$$H(h, b_h) = 1{\cdot}0$$

$$H(\phi, \phi) = 1{\cdot}0$$

$$H(\phi, b_\phi) = 1{\cdot}0$$

$$H(\theta, \theta) = 1{\cdot}0$$

$$H(\theta, b_\theta) = 1{\cdot}0$$

$$H(\psi, \psi) = 1{\cdot}0$$

$$H(\psi, b_\psi) = 1{\cdot}0$$

The coefficient of the input matrix in Eqn (B.1) are given below. The arguments in the notation $B(i,j)$ denote the state equation and input variable respectively.

$$B(u, a_x) = -1{\cdot}0$$

$$B(u, q) = w$$

$$B(u, r) = -v$$

$$B(v, a_y) = -1{\cdot}0$$

$$B(v, p) = -w$$

$$B(v, r) = u$$

$$B(w, a_z) = 1{\cdot}0$$

$$B(w, p) = v$$

$$B(w, q) = -u$$

$$B(\phi, p) = 1 \cdot 0$$
$$B(\phi, q) = \sin\phi \tan\theta$$
$$B(\phi, r) = \cos\phi \tan\theta$$

$$B(\theta, q) = \cos\phi$$
$$B(\theta, r) = -\sin\phi$$

$$B(\psi, q) = \sin\phi / \cos\theta$$
$$B(\psi, r) = \cos\phi / \cos\theta$$

APPENDIX C

Example of Computer Program Set Up Data

.RU CHOICE

PROCESS NOISE VARIANCE

QUANTITY	NUMBER	VALUE
NX	1	0·00010000
NY	2	0·00160000
NZ	3	1·00000000
P	4	0·00010000
Q	5	0·00010000
R	6	0·00010000

MEASUREMENT NOISE VARIANCE

VFP	7	1·00000000
BETA	8	0·00002500
ALPHA	9	0·00002500
HEGHT	10	1000·00000000
PHI	11	0·01000000
THETA	12	0·00250000
PSI	13	0·01000000

TO CHANGE VALUE TYPE I,F
TERMINATE CHANGES WITH 0,0

TYPE −1,0,1 TO REVIEW, TERMINATE, OR VIEW CHANNELS

I

INPUT MEASUREMENTS AVAILABLE

QUANTITY	NUMBER	STATUS	I/C IF NOT MEASURED
NX	1	1	0·00000000
NY	2	2	0·00000000
NZ	3	3	0·00000000
P	4	4	0·00000000
Q	5	5	0·00000000
R	6	6	0·00000000

TO CHANGE STATUS TYPE I,I ADD −1,F TO CHANGE I/C
TERMINATE CHANGES WITH 0,0

MEASUREMENTS TO BE INCLUDED

QUANTITY	NUMBER	STATUS	I/C IF NOT MEASURED
VFP	1	1	0·00000000
BETA	2	2	0·00000000
ALPHA	3	3	0·00000000
HEGHT	4	4	0·00000000
PHI	5	5	0·00000000
THETA	6	6	0·00000000
PSI	7	7	0·00000000

TO CHANGE STATUS TYPE I,I ADD −1,F TO CHANGE I/C
TERMINATE CHANGES WITH 0,0

STATES TO BE ESTIMATED

QUANTITY	NUMBER	STATUS	STATES WHICH CAN BE ESTIMATED	VARIANCE
CHANGE AS BEFORE ADD −1,F TO CHANGE VARIANCE				
U	1	1	1·0	4·00000000
V	2	2	1·0	0·00250000
W	3	3	1·0	0·01000000
HEGHT	4	4	1·0	100·00000000
PHI	5	5	1·0	0·00002500
THETA	6	6	1·0	0·00000400
PSI	7	7	1·0	0·00000400
BAX	8	0	1·0	0·07700000
BAY	9	0	1·0	0·07700000
BAZ	10	0	1·0	0·07700000
BP	11	0	1·0	0·02286000
BQ	12	0	1·0	0·01940000
BR	13	0	1·0	0·07700000
BVFP	14	0	1·0	0·07700000
BBETA	15	0	1·0	0·07700000
BALFA	16	0	1·0	0·07700000
BHGHT	17	0	1·0	0·07700000
BPHI	18	0	1·0	0·07700000
BTHTA	19	0	1·0	0·07700000
BPSI	20	0	1·0	0·07700000
KVFP	21	0	1·0	0·07700000
KBETA	22	0	1·0	0·07700000
KALFA	23	0	1·0	0·07700000

TYPE 1 OR 0 IF REPEAT IS/IS NOT REQUIRED

PROGRAM STEPS	OPERATION
[1] Initialise program	Prepared by program 'CHOICE'
[1.1] Read in set up data	
[1.2] Read initial measurements	
[2] Initialise state estimates	$\left[\hat{X}^{+}(k)\right] = \left[\hat{X}(o)\right]$
[3] Initialise state covariance estimates	$S^{+}(k) = S(o)$
[4] Start filter	
[5] Output states and state variances	
[6] Calculate input matrix	$\left[B\ (k+1)\right]$ and $\left[\Gamma(k+1)\right]$
[7] State prediction by numerical integration of nonlinear equations	$\left[\hat{X}\ (k+1)\right] = \int_{k\Delta t}^{(k+1)\Delta t} f(\hat{X}^{+}(k), \eta, \Theta)\,dt$
[8] Calculation of state transition matrix by power series approximation	$\left[\Phi(k+1, k)\right] = e^{A\Delta t}$
[9] Covariance prediction	$W(k+1) = \Phi(k+1,k)\ S^{+}(k)$ $S\ (k+1) = \left[W(k+1)\ W(k+1)^{T} + Q(k+1)\right]^{1/2}$
Matrix partitioning	$\left[\begin{array}{c:c} \Phi_{11} & O \\ \hdashline O & \Phi_{22} = I \end{array}\right] \times \left[\begin{array}{c:c} S^{+}(k)_{11} & S^{+}(k)_{12} \\ \hdashline & S^{+}(k)_{22} \end{array}\right]$

FIG. 1 SUMMARY OF PROGRAM STEPS

PROGRAM STEPS	OPERATION
9.1 $\Phi_{11} \times S^{+}(k)_{11/12} = W(k+1)_{11/12}$	
9.2 $W(k+1)_{12} \rightarrow P(k+1)_{12}$	
9.3 $W(k+1) \times W(k+1)^{T} = P(k+1)_{11}$	
9.4 $S^{+}(k)_{22} \rightarrow P(k+1)_{22}$	
9.5 Calculation of process noise covariance matrix and formation of $P(k+1)_{11/12/22} + Q(k+1) = P(k+1)$	
9.6 CHOLESKI decomposition	
10 Read in new measurements	

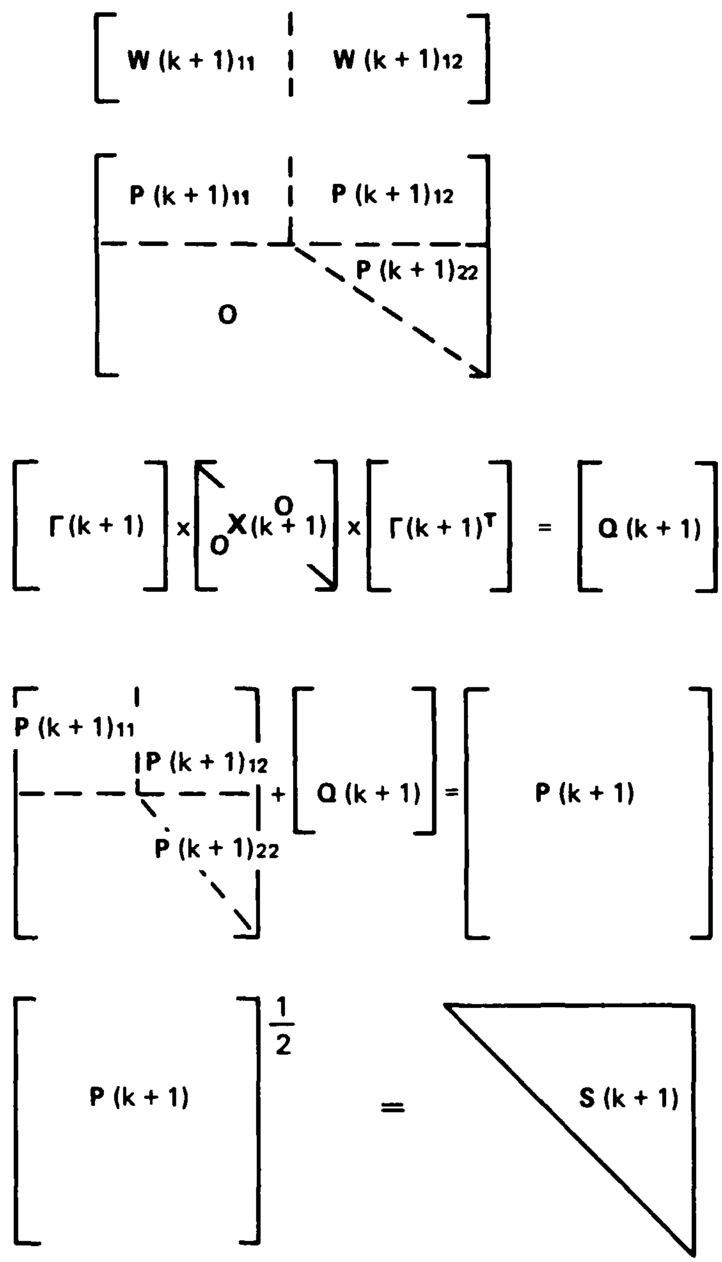
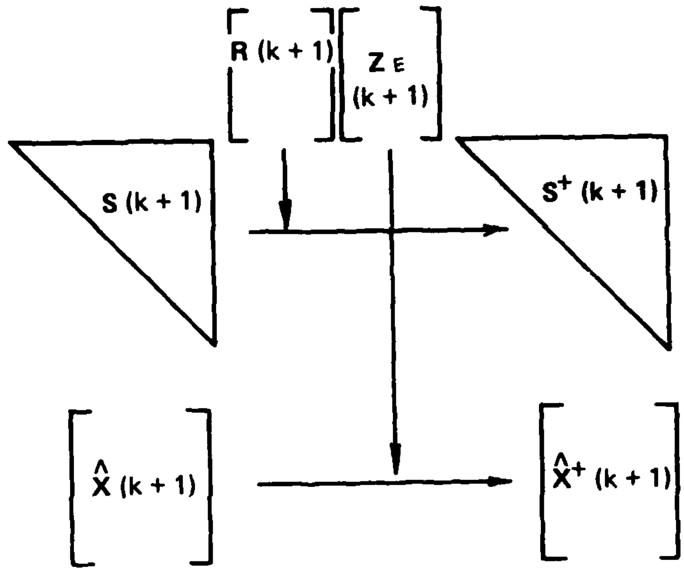


FIG. 1 CONTINUED

PROGRAM STEPS

OPERATION

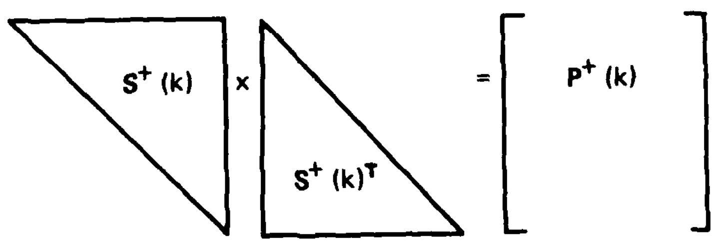
[11] *Correction of covariance and state matrices*
$S(k+1)$ and $X(k+1)$ using CARLSON algorithm



[12] Next data point
$S^{+}(K) | = S^{+}(k+1)$
$\hat{X}^{+}(k) \quad \hat{X}^{+}(k+1)$

Return to step 5

[13] Calculate and output bias errors



[14] Calculate and output state variances

End

FIG. 1 CONTINUED

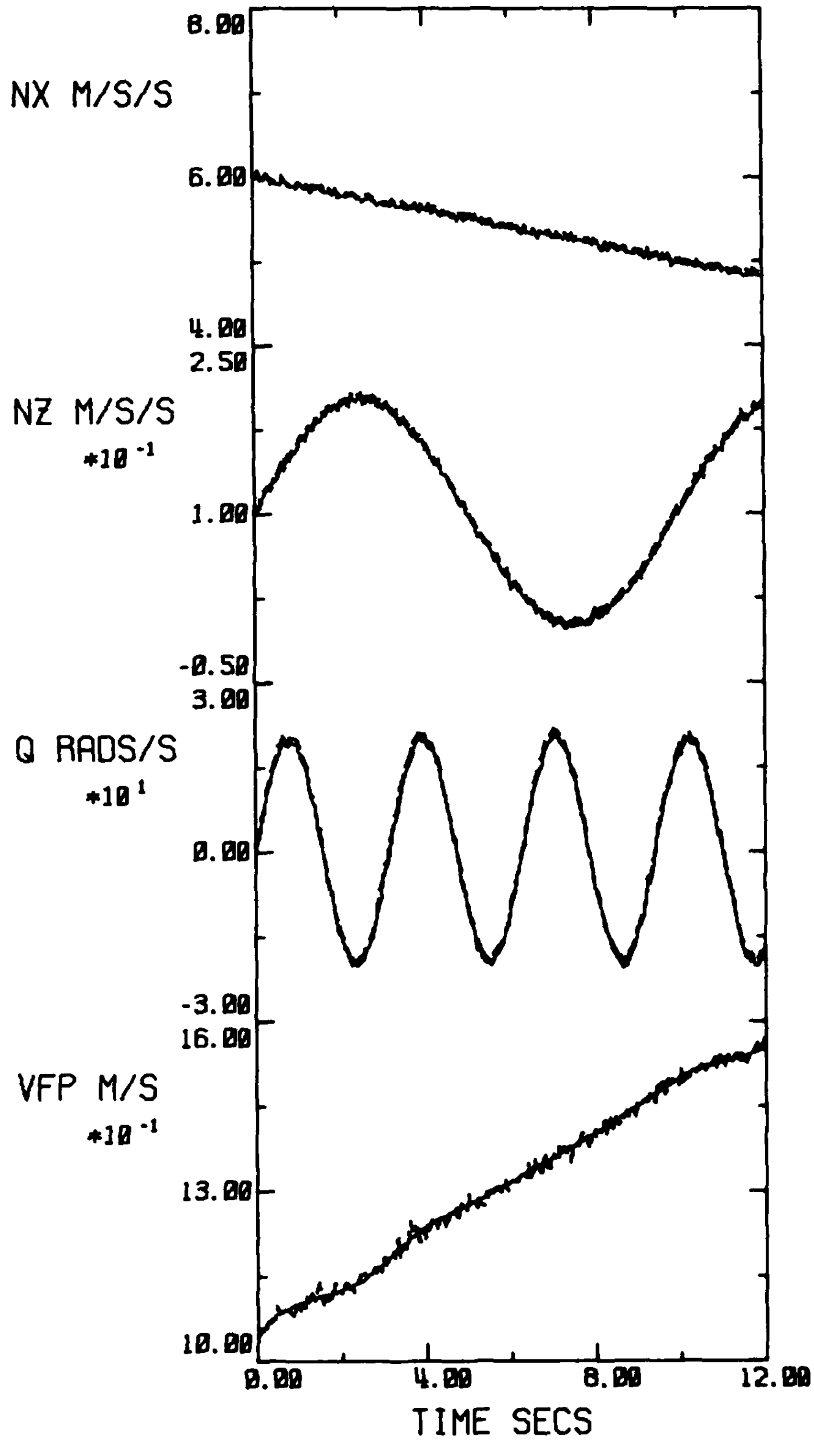
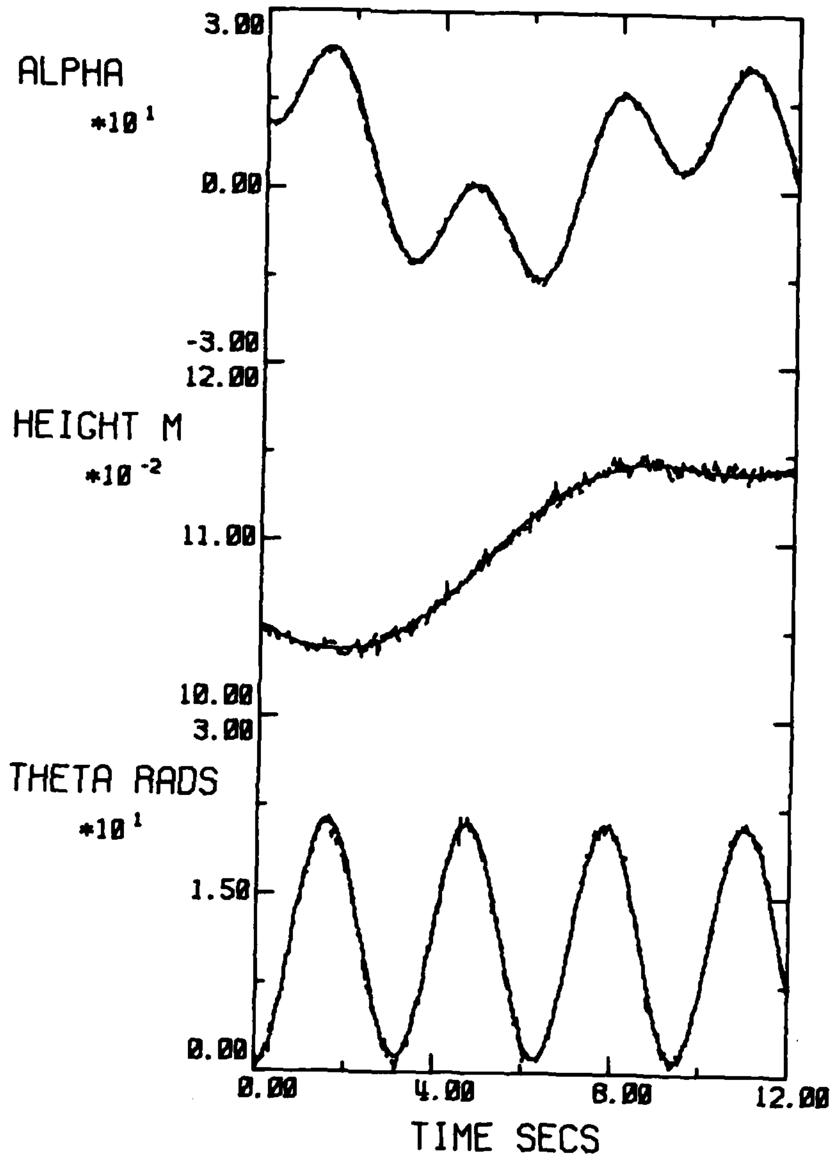


FIG. 2 COMPATIBLE DATA



ALPHA
*10 1
3.00
0.00
-3.00
HEIGHT M
*10 -2
12.00
11.00
10.00
THETA RADS
*10 1
3.00
1.50
0.00
0.00
4.00
8.00
12.00
TIME SECS

FIG. 2 (CONT.)

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